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HARMONIC ANALYSIS ON NON-SEMISIMPLE SYMMETRIC SPACES

BY
RONALD L. LIPSMAN†
Department of Mathematics, University of Maryland, College Park, MD 20742, USA

ABSTRACT
It is proven that the L^2 spectrum for certain non-semisimple, non-nilpotent symmetric spaces is multiplicity-free. The spectrum and spectral measure are computed precisely for symmetric spaces corresponding to non-compact motion groups. Somewhat less complete results on the L^2 spectrum — in both the Mackey Machine and Orbit Method modes — are given for general semidirect product symmetric spaces.

1. Introduction

Let G/H be a symmetric space, that is, G is a Lie group and $H = H^\sigma$ is the stabilizer of an involutive automorphism σ of G. The quasi-regular representation $R = R_{G,H}$ of G on $L^2(G/H)$ is an object that has commanded tremendous amounts of attention — especially in the case that G is semisimple. Recently there has been an upsurge of interest in non-semisimple symmetric spaces. Much of this has evolved out of the active study of solvability (and other properties) of invariant differential operators on homogeneous spaces. The usual questions are addressed: (1) when is $L^2(G/H)$ multiplicity-free; and if so (2) describe as explicitly as possible the spectrum and spectral measure. As one might expect, most of the analysis has been confined to the case of nilpotent groups. A prime example is Benoist's result that the spectrum is multiplicity-free (valid when G is exponential solvable). It is my purpose in this paper to present some results on the spectrum for non-nilpotent, non-semisimple symmetric spaces. It is well-known that in such a situation, the spectrum may not be multiplicity-free. Here I utilize Benoist's scenario to specify a class of spaces for which the spectrum is

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multiplicity-free. I also study the more difficult problem of actually describing (explicitly) the contents of the spectrum and the spectral measure class.

The paper is organized into four sections in addition to the introduction. In the second section I generalize the classical situation of a symmetric space associated to a motion group to allow a "non-compact group of rotations" (Theorem 2.1). The arguments are essentially classical, but the number of examples is enormous. This allows me to illustrate that there are myriad possibilities for the spectrum of a symmetric space. I organize the question of the composition of the spectrum around the idea of how closely it resembles that of the regular representation, and I enumerate examples to illustrate the various possibilities. In section 3, I apply Benoist's criterion to semidirect product spaces. Basically, I look at $G = KS$ where S is exponential solvable and normal, σ is an involutive automorphism of S that is stabilized by K (which is otherwise arbitrary) and $H = KS^\sigma$. I show that $L^2(G/H)$ is multiplicity-free (Theorem 3.4). The most interesting example is when S is the nilradical of an arbitrary parabolic group, K is its Levi component, and σ is the canonical involution (which I define in Example 3.5 (iii)). In section 4 the issue of explicit information on the spectrum is taken up. The preliminary result — which is folklore — is that the spectrum is contained in

$$\{\pi \in \hat{G} : \pi^\sigma \cong \bar{\pi}\}. \tag{1.1}$$

Indeed it may actually be smaller. But in the nilpotent case, Benoist cites work of Grelaud to conclude that (1.1) is the precise spectrum. He also gives an orbital description of the spectrum and the spectral measure. We give three results. One is a uniqueness theorem which relates the spectrum of $L^2(G/H)$ to that of $L^2(S/S^\sigma)$ (Theorem 4.1); a second shows that no Mackey obstruction may enter into the group extension representations that occur in the spectrum (Proposition 4.3); and the third is a partial result on a generalization of Benoist's orbital description of the spectrum (Proposition 4.6). Finally, in section 5 we supply two further pieces of information. The first is a negative answer to a question of Corwin which asked if a multiplicity-free representation quasi-equivalent to the regular representation could always be constructed from a symmetric space in the nilpotent case. The second is an introductory comment on a specific problem in the general study of spectral theory for *non*-symmetric homogeneous spaces.

2. Abelian symmetric spaces

The classic situation of a symmetric space associated to a motion group is the following. Let $G = KV$ be a semidirect product of an abelian normal subgroup

V with a compact group K. The description of the spectrum of $L^2(G/K)$ is familiar. The irreducible unitary representations of G are parameterized by the Mackey Machine [12]:

$$\pi_{\chi,\tau} = \mathrm{Ind}_{K_\chi V}^G \tau\chi, \qquad \chi \in \hat{V}, \quad \tau \in \hat{K}_\chi.$$

Furthermore we have

$$R_{G,K} = \int_{\hat{V}/K}^{\oplus} \pi_{\chi,1} d\bar{\chi} \tag{2.1}$$

where $d\bar{\chi}$ is the image of Haar measure $d\chi$ on $\hat{V}$ under the projection $\hat{V} \to \hat{V}/K$.

In fact the compactness of K plays a limited role (specified momentarily) in the above. Let $G = HV$ be a semidirect product of an abelian normal subgroup V with any separable locally compact group H. If we assume $\hat{V}/H$ is countably separated we can still obtain irreducible representations of G via the Mackey Machine [12]:

$$\pi_{\chi,\tau} = \mathrm{Ind}_{H_\chi V}^G \tau\chi, \qquad \chi \in \hat{V}, \quad \tau \in \hat{H}_\chi.$$

THEOREM 2.1. *Suppose $\hat{V}/H$ is countably separated. Then $R_{G/H}$ is multiplicity-free and the spectrum is described by*

$$R_{G,H} = \int_{\hat{V}/H}^{\oplus} \pi_{\chi,1} d\bar{\chi}$$

where $d\bar{\chi}$ denotes any pseudo-image of Haar measure $d\chi$ on $\hat{V}$ under the natural projection $\hat{V} \to \hat{V}/H$.

NOTES. (i) We see that the only role of compactness is to guarantee the smoothness of $\hat{V}/H$ and that we can take the image measure instead of the pseudo-image.

(ii) It goes virtually without saying that the involutive automorphism of G is $\sigma(hv) = hv^{-1}$, $h \in H$, $v \in V$.

PROOF. We use disintegration of measures. Fix Haar measure dv on V and let $d\chi$ be the dual measure. We write $\delta(h)$ for the modulus of the automorphism $v \to h \cdot v = hvh^{-1}$ on V. Having chosen a pseudo-image $d\bar{\chi}$, there are uniquely determined relatively invariant measures $d\mu_\chi$ on H/H_χ such that

$$\int_{\hat{V}} f(\chi)d\chi = \int_{\hat{V}/H}\int_{H/H_\chi} f(h \cdot \chi)d\mu_\chi(h)d\bar{\chi}.$$

Combining this with the Fourier transform $\mathcal{F}$ on V, we have an intertwining operator

$$L^2(G/H) \cong L^2(V) \xrightarrow{\mathcal{F}} L^2(\hat{V}) \cong \int_{\hat{V}}^{\oplus} \mathbf{C}_\chi d\chi \cong \int_{\hat{V}/H_\chi}^{\oplus} \int_{H/H_\chi}^{\oplus} \mathbf{C}_{h\cdot\chi} d\mu_\chi d\bar{\chi}.$$

The action of $R = R_{G,H}$ on $L^2(\hat{V})$ is

$$R(hv)f(\chi) = h^{-1}\cdot\chi(v)f(h^{-1}\cdot\chi)\delta^{1/2}(h). \tag{2.2}$$

But the space

$$\int_{H/H_\chi}^{\oplus} \mathbf{C}_{h\cdot\chi} d\mu_\chi$$

is actually G-invariant (this is obvious). If we denote the representation of G thus obtained by $\pi(\chi)$, then the preceding proves that

$$R = \int_{\hat{V}/H}^{\oplus} \pi(\chi) d\bar{\chi}.$$

But it is easy to identify $\pi(\chi)$ with $\pi_{\chi,1}$. Indeed the representation $\pi_{\chi,1} = \mathrm{Ind}_{H_\chi V}^G \chi$ may be realized in the space $L^2(H/H_\chi)$ according to the rule

$$\pi_{\chi,1}(hv)f(yH_\chi) = h^{-1}\cdot y\cdot\chi(v)f(h^{-1}yH_\chi)\delta(h)^{1/2}.$$

When we identify $L^2(H/H_\chi) = \int_{H/H_\chi}^{\oplus} \mathbf{C}_{h\cdot\chi} d\mu_\chi(h)$ the action becomes exactly that of (2.2), i.e. of $\pi(\chi)$. The proof is completed by invoking the Mackey Machine to assert that $\pi_{\chi,1} \not\cong \pi_{\chi',1}$ if χ and χ' are in different H-orbits. (Note we have used the fact that the modulus of the relatively invariant measure on H/H_χ agrees a.a. with δ — see [6, II].)

One of the most interesting questions one can ask about the decomposition of $R_{G,H}$ in Theorem 2.1 is to what extent it mirrors that of the regular representation. The irreducibles in the regular representation occur with multiplicity equal to their dimension. A common problem these days is to find a natural or canonical construction of a multiplicity-free representation, quasi-equivalent to the regular representation. It is natural to employ symmetric spaces in this search. So how close to the Plancherel spectrum is the spectrum of $R_{G,H}$? In order to facilitate our discussion of this question — both here and for more general symmetric spaces — we introduce some new terminology.

DEFINITION 2.2. Let G be a Lie group, H the stabilizer of an involutive automorphism of G. We say that $R_{G,H}$ is:

(a) *hefty* if $R_{G,H} \approx R_G$;

(b) *average* if $R_{G,H} \underset{\approx}{\subseteq} R_G$;

(c) *slim* if $R_{G,H}$ is quasi-equivalent to a representation weakly contained in R_G, but no subrepresentation of $R_{G,H} \subsetapprox R_G$;

(d) *unusual* if none of the above obtains.

(Sometimes we abuse terminology by saying that G/H or H itself is hefty, etc.) Here: $\approx$ means quasi-equivalent and $\subsetapprox$ means quasi-equivalent to a subrepresentation. If G is type I and we denote the (essentially unique) decomposition of a unitary representation T of G by $T = \int_{\hat{G}}^{\oplus} n_T(\pi)\pi d\mu_T(\pi)$, then: (a) means the measure classes of $\mu_{R_{G,H}}$ and μ_{R_G} are the same; (b) means $\mu_{R_{G,H}}$ is absolutely continuous with respect to μ_{R_G}; and (c) means $\operatorname{Supp} R_{G,H} \subseteqq \operatorname{Supp} R_G$ but $\mu_{R_G}(\operatorname{Supp} R_{G,H}) = 0$. The unusual case can arise for two reasons — either an open subset of $\operatorname{Supp} R_{G,H}$ is disjoint from $\operatorname{Supp} R_G$; or $\operatorname{Supp} R_{G,H} \subseteqq \operatorname{Supp} R_G$ but part of the spectrum is average and part is slim. Hopefully the following will help to clarify these ideas.

For Riemannian symmetric spaces G/K, $R_{G,K}$ is average if G has finite center, but otherwise slim. However non-Riemannian semisimple symmetric spaces may be unusual. As for abelian symmetric spaces, all possibilities (a)–(d) can occur. First if H is compact, only (a) and (b) are possible. For example, in the symmetric space G/H corresponding to the Euclidean motion group $G = \mathrm{SO}(n)\cdot\mathbf{R}^n$, $H = \mathrm{SO}(n)$, H is average when $n > 2$, but hefty when $n = 2$. If we consider instead the symmetric spaces G/H corresponding to the affine motion group $G = \mathrm{GL}(n,\mathbf{R})\cdot\mathbf{R}^n$, $H = \mathrm{GL}(n,\mathbf{R})$, then H is slim unless $n = 1$ in which case it is hefty. The symmetric space G/H where $G = \mathrm{SO}_e(n,1)\cdot\mathbf{R}^{n+1}$, $H = \mathrm{SO}_e(n,1)$ is unusual because it cannot decide between slim and average. But if we replace the Lorentz group by symplectic groups $\mathrm{Sp}(n,1)$, then the situation becomes unusual because of the support condition.

Let us conclude by noting that in general $R_{G,H}$ is hefty (resp. average) iff H_χ is trivial (resp. compact) for almost all $\chi \in \hat{V}$. Finally $R_{G,H}$ is irreducible iff H is essentially transitive on $\hat{V}$.

3. The Benoist scenario

As always G is a Lie group and H is the stabilizer of an involutive automorphism σ of G. The following definition is from [1].

DEFINITION 3.1. We say H has property (MF) if there is a submanifold Q in G such that

(i) the multiplication map $m : H \times Q \to G$ is a surjective submersion;

(ii) $Q^{-1} = Q$;

(iii) $\forall q \in Q$, $\sigma(q)q \in H$;

(iv) $\forall h \in H$, $hQh^{-1} = Q$.

(Benoist says H has the property $\mathscr{P}$.)

EXAMPLE 3.2. If G is exponential solvable, then the stabilizer of *any* involutive automorphism σ has property (MF). Benoist [1] proves this with $Q = \{g \in G : \sigma(g) = g^{-1}\}$.

Benoist then proves

THEOREM 3.3. *If for the symmetric space G/H, the subgroup H has property* (MF), *then $R_{G,H}$ is multiplicity-free.*

Now we place ourselves in the following situation. S is a Lie group with involutive automorphism σ such that the stability group $T = S^\sigma$ has property (MF). According to Theorem 3.3, $R_{S,T}$ is multiplicity-free. We assume that a Lie group K acts on S by automorphisms, giving a semidirect product $G = KS$. We assume further that K preserves σ, that is

$$k \cdot \sigma(s) = \sigma(k \cdot s), \qquad k \in K, \quad s \in S.$$

In particular K preserves T. We assume finally that K preserves a submanifold Q of S with respect to which T has property (MF). (This last assumption will hold automatically in the cases of interest we have in mind.)

THEOREM 3.4. *The subgroup $H = KT$ has property* (MF) *in G for the involution*

$$\sigma(ks) = k\sigma(s), \qquad k \in K, \quad s \in S.$$

Therefore $R_{G,H}$ is multiplicity-free.

PROOF. According to our assumptions there is a submanifold Q in S which satisfies properties (i)–(iv) of Definition 3.1, and which is K-invariant. In order to demonstrate the theorem, it is enough to verify that Q satisfies the same four properties with respect to $H = KT$.

(i) Since $m : T \times Q \to S$ is a surjective submersion and $G = K \cdot S$, it is obvious that the multiplication map $m : KT \times Q \to G$ is also a surjective submersion.

(ii) $Q^{-1} = Q$ by hypothesis.

(iii) $\forall q \in Q$, we have $\sigma(q)q \in T \subseteq H$.

(iv) If $h = kt \in H$, then $hQh^{-1} = ktQt^{-1}k^{-1} = kQk^{-1} = k \cdot Q \subseteq Q$. The latter is true $\forall t \in T$, $\forall k \in K$. In particular $k^{-1} \cdot Q \subseteq Q$, and so $h \cdot Q = k \cdot Q = Q$.

The extension from Benoist's Theorem 3.3 to our Theorem 3.4 is not difficult. However it is interesting for two reasons. First of all there are many nice examples to which it applies — a few of which we describe momentarily. Second, as with Theorem 3.3, it leaves to the imagination the description of the precise spectrum of $R_{G,H}$. Even in the case of exponential solvable groups the identification of the precise spectrum is a delicate matter — not really settled by Benoist (see also Remark 3.7(ii)). But even if we assume that the spectrum of $L^2(S/T)$ is known, we are left with the interesting problem of describing that of $L^2(KS/KT)$. What could it be? Since we are dealing with semidirect products $G = KS$ with K essentially arbitrary, the most reasonable description in general of the representations of G which occur in $R_{G,H}$ should be through the Mackey Machine. The candidates are $\pi_{\gamma,\tau} = \mathrm{Ind}_{G_\gamma}^{G}\tau$, $\gamma \in \hat{S}$ in Supp $R_{S,T}$, $\tau \in \hat{G}_\gamma$ with $\tau|_S$ a multiple of γ. The work of section 2 suggests that for each γ, there exists a *unique* τ such that $\pi_{\gamma,\tau}$ is in the spectrum (see Theorem 4.1). But what might it be? Even though G_γ splits $G_\gamma = K_\gamma S$, and even if the obstruction to extending γ to a representation of $K_\gamma S$ is trivial (i.e. there is no Mackey co-cycle — see Proposition 4.3), there is still an ambiguity. Namely the extension of γ to a representation τ of $K_\gamma S$ is only unique up to an arbitrary unitary character of K_γ. How do we specify it?

We shall deal with these interesting questions in section 4. In the remainder of this section we supply examples to illustrate Theorem 3.4. As in section 2 we examine various possibilities in terms of the relation of $R_{G,H}$ to the regular represenation of G. All unexplained claims about spectra are consequences of the results in section 4.

EXAMPLES 3.5. (i) If S is exponential solvable, then we may take $Q = \{s \in S : \sigma(s) = s^{-1}\}$. If we have an action by a group K which preserves σ, then the submanifold Q is automatically K-invariant. We shall deal essentially only with this case in the paper.

(ii) (Abelian symmetric spaces). If S is abelian and $\sigma(s) = s^{-1}$, then any group K of automorphisms of S preserves σ. We obtain the spaces of section 2.

(iii) (Canonical involution in parabolic groups). Let P be a parabolic subgroup of a reductive Lie group G of the Harish–Chandra class. Suppose $P = MAN$ is its Langlands decomposition. We denote by Δ the (restricted) roots of $(\mathfrak{g}, \mathfrak{a})$, and by Δ^+ the positive roots determined by N, i.e. $\Delta^+ = \{\alpha \in \Delta : \mathfrak{g}^\alpha \subseteq \mathfrak{n}\}$. We denote the simple roots in Δ^+ by $\Sigma = \{\alpha_1, \dots, \alpha_r\}$. Every $\alpha \in \Delta^+$ may be written uniquely $\alpha = \Sigma_{i=1}^r n_i\alpha_i$, $n_i \geqq 0$. We define a canonical involution ρ on N as follows:

For any $\alpha \in \Delta^+$, we set $\rho|_{\mathfrak{g}^\alpha} = \pm\mathrm{Id}$ according as $\Sigma_{i=1}^r n_i$ is even or odd, ρ is extended to $\mathfrak{n} = \Sigma_{\alpha\in\Delta^+}\mathfrak{g}^\alpha$ by linearity, and to N by exponentiation.

It is fairly routine to check that ρ is an involutive automorphism. Now the key point is that ρ is invariant under the action of MA. This is simply because MA preserves each root space. Thus we are in the situation of Theorem 3.4 if we take $S = N$ and K any subgroup of MA. When $K = MA$ we call the extended involution ρ the *canonical involution on the parabolic P*.

It is clear that we have an enormous number of examples — the most interesting of which occur with the choice of K to be M, A, MA or trivial. We illustrate the possibilities by examining several of these in the context of Definition 2.2.

(a) Consider first the (minimal) parabolic subgroup of the split rank one semisimple Lie group $G = \mathrm{SO}_e(n+1,1)$, $n \geqq 2$. Then

$$P = MAN \cong \mathrm{SO}(n) \times \mathbf{R}^+ \cdot \mathbf{R}^n.$$

The nilradical is abelian, $\rho = -\mathrm{Id}$ and $N^\rho = \{1\}$. Of course N/N^ρ is hefty. But we also have A is hefty in AN, i.e. $R_{AN,A} \approx R_{AN}$. On the other hand M is average in MN and MA is average in P (when $n > 2$). More precisely, the generic representations of P are parameterized as follows. Fixing any $\chi \in \hat{N}$, $\chi \neq 1$, we have

$$\pi_\tau = \mathrm{Ind}_{M_1N}^{P} \tau\chi, \qquad \tau \in \hat{M}_1, \quad M_1 = M_\chi \cong \mathrm{SO}(n-1).$$

Then $R_{P,MA} \cong \pi_1$ is actually irreducible. (If $n = 2$ it is also hefty.)

(b) Next consider the split rank one group $G = \mathrm{SU}(n,1)$. Then the minimal parabolic $P = MAN$ has a $(2n+1)$-dimensional Heisenberg group for nilradical, $N^\rho \cong Z_N$, A is a group of positive reals and M is a (slightly twisted) compact unitary group. All of the symmetric spaces KN/KN^ρ, $K = M, MA$, etc. are slim. The same situation obtains with the other rank one simple groups.

(c) Now let us examine the symplectic group $G = \mathrm{Sp}(2,\mathbf{R})$. A minimal parabolic subgroup $P = MAN$ of G has nilradical isomorphic to the well-known 3-step 4-dimensional nilpotent Lie group, the group A is a direct product of two positive real lines and M is finite. If we write $\Sigma = \{\alpha, \beta\}$, then $\mathfrak{n} = \mathfrak{g}_\alpha + \mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{2\alpha+\beta}$ and $N^\rho = \exp \mathfrak{g}_{\alpha+\beta}$. In fact N^ρ is hefty in N and AN^ρ is hefty in AN.

(d) Here is a general result about parabolics with abelian nilradical. To demonstrate it, we first state another general result for arbitrary parabolics.

PROPOSITION 3.6. *The class of the representation* $\mathrm{Ind}_A^{AN}\tau$ *is independent of* $\tau \in \hat{A}$.

This can be proven by generalizing the argument of [8, Thm. 3], but I omit the rather long details here. Now suppose N is abelian. Then $N^{\rho} = \{1\}$, and it follows that A is hefty in AN. This is because

$$\mathrm{Ind}_{\{1\}}^{AN}1 \cong \mathrm{Ind}_A^{AN}\,\mathrm{Ind}_{\{1\}}^{A}1 = \mathrm{Ind}_A^{AN}\int_{\hat{A}}^{\oplus} \tau d\tau = \int_{\hat{A}}^{\oplus} \mathrm{Ind}_A^{AN}\tau d\tau \cong \infty\,\mathrm{Ind}_A^{AN}1.$$

That is, $R_{AN,A} \approx R_{AN}$.

REMARKS 3.7. (i) We leave to the reader the pleasure of working out other examples — for example, a nice one is the minimal parabolic in $\mathrm{SL}(4,\mathbf{R})$. For specific groups of low enough dimension, the precise spectrum can be obtained by a combination of the techniques of section 4, Mackey or Anh reciprocity [7, II.A.4], Proposition 3.6 and the use of bare hands.

(ii) For more general situations or general classes of semidirect product symmetric spaces, the actual determination of the spectrum is a difficult job — as it is with any class of non-abelian symmetric spaces G/H. Reciprocity suggests one should look among the irreducibles of G which contain an H-fixed vector. For nilpotent, or more generally exponentially solvable, symmetric spaces S/T, such reciprocity suggests the following for the spectrum of $L^2(S/T)$ in $\hat{S}$. It should be parameterized via the Kirillov map by $\mathfrak{t}^{\perp}/T$, $\mathfrak{t}^{\perp} = \{\theta \in \mathfrak{s}^* : \theta(\mathfrak{t}) = 0\}$, and the spectral measure should be a pseudo-image of Lebesgue measure. Benoist obtains this result for the nilpotent case as a consequence of [5]. In general it is not known. We shall say more about this in the next section.

(iii) The setup of Theorem 3.4 may appear excessively simple. Conceivably there could be involutive automorphisms on a semidirect product $G = KS$ which are more elaborate on K. But in fact that is almost impossible to arrange. It is very difficult to obtain an involutive automorphism σ on $G = KS$ with σ acting non-trivially on K unless the product is direct or $\sigma|_S$ is virtually trivial. (We do not have a precise result.)

4. Results on the spectrum

Given an exponential solvable symmetric space S/T, or a semidirect product extension KS/KT, we know that the corresponding quasi-regular representation is multiplicity-free. The problem then is to describe as explicitly as possible the spectrum. I have already discussed the following ideas: in general the spectrum is contained in but may not equal the set $\{\pi \in \hat{G} : \pi^{\sigma} \cong \bar{\pi}\}$; in the exponential solvable case the latter is naturally isomorphic to $\mathfrak{t}^{\perp}/T$; by involving more general disintegration work, Benoist has concluded that this is the precise spectrum in the nilpotent case. If one can believe the results of [2], it would also be the spectrum in the exponential solvable case. But no proof of the results in [2] have appeared; and I know of no independent corroboration.

In this section the general approach we take is as follows. We assume that the spectrum of $L^2(S/T)$ is known — can we derive the spectrum of $L^2(KS/KT)$? We give several results addressed to this problem — most from the perspective of the Mackey Machine, one from an orbital perspective.

As in section 3 the picture is: S/T is symmetric, T has property (MF) with respect to σ, and K is a group of automorphisms of S preserving all relevant data. We set $G = KS$, $H = KT$. (We often assume S is exponential solvable.) We make three further assumptions — namely S is type I, $\hat{S}/K$ is countably separated and G is type I. Then the irreducible unitary representations of G are parameterized by $\pi = \pi_{\gamma,\tau} = \mathrm{Ind}_{G_\gamma}^G \tau$, $\gamma \in \hat{S}$, $G_\gamma =$ the stability group, $\tau \in \check{G}_\gamma = \{\tau \in \hat{G}_\gamma : \tau|_S = \text{multiple of } \gamma\}$. We denote by $\mathcal{S} = \mathcal{S}_{S,T}$ the spectrum of $R_{S,T}$, i.e. the smallest closed subset of $\hat{S}$ which supports the spectral measure (class) $\mu_{S,T}$ of $R_{S,T}$. So according to Theorem 3.3 we have

$$R_{S,T} = \int_{\mathcal{S}_{S,T}}^{\oplus} \gamma d\mu_{S,T}(\gamma), \tag{4.1}$$

a multiplicity-free decomposition.

THEOREM 4.1. *For $\mu_{S,T}$-almost all $\gamma \in \hat{S}$ there exists a unique irreducible unitary representation $\tau = \tau(\gamma)$ of $G_\gamma = K_\gamma S$, whose restriction to S* equals γ, *such that*

$$R_{G,H} = \int_{\mathcal{S}_{G,H}}^{\oplus} \pi_{\gamma,\tau(\gamma)} d\mu_{G,H}(\pi).$$

Furthermore $\mu_{G,H}$ is a pseudo-image of $\mu_{S,T}$ under the map $\hat{S} \to \hat{S}/K$.

PROOF. The representation $R_{G,H}$ acts in $L^2(G/H) \cong L^2(S/T)$, so in principle we could use the same idea as in the proof of Theorem 2.1. However, unlike the abelian case, we do not have the explicit intertwining operator that effects the disintegration (4.1). Instead we reason as follows. We know by Theorem 3.4 that

$$R_{G,H} = \int_{\mathcal{S}_{G,H}}^{\oplus} \pi_{\gamma,\tau} d\mu_{G,H}(\gamma, \tau) \tag{4.2}$$

for some measure class $\mu_{G,H}$. Restrict both sides to the subgroup S. We obtain

$$R_{S,T} = \int_{\mathcal{S}_{G,H}}^{\oplus} \pi_{\gamma,\tau}|_S d\mu_{G,H}(\gamma, \tau). \tag{4.3}$$

But the decomposition $\pi_{\gamma,\tau}\big|_S$ is well-known (see e.g. [6]). In fact

$$\pi_{\gamma,\tau}\big|_S = \int_{K/K_\gamma}^{\oplus} k\cdot\gamma\otimes 1_{\dim\tau/\dim\gamma}\,d\mu_\gamma(k), \tag{4.4}$$

for the (relatively) invariant measure class μ_γ on K/K_γ. Now by examining equations (4.1)–(4.4) we can conclude the following facts:

(i) The representations that occur in the support of $R_{G,H}$ must lie over $\mathcal{S}_{S,T}$.

(ii) For $\mu_{S,T}$-a.a. $\gamma\in\hat{S}$, we must have $\dim\tau=\dim\gamma$ for any $\tau\in\check{G}_\gamma$ such that $\pi_{\gamma,\tau}\in\mathcal{S}_{G,H}$. That is, γ extends to a representation $\tilde{\gamma}$ of G_γ in the space of γ; and the only possibilities for other such extensions are $\tilde{\gamma}\otimes\chi$, χ a character of $G_\gamma/S=K_\gamma$.

(iii) Since $\pi_{\gamma,\tau}\big|_S\cong\pi_{\gamma,\tau\otimes\chi}\big|_S$ for any character χ, by the multiplicity-free property of $R_{S,T}$, we must have for $\mu_{S,T}$-a.a. $\gamma\in\hat{S}$, a unique choice of $\tau\in\check{G}_\gamma$ such that $\pi_{\gamma,\tau}\in\mathcal{S}_{G,H}$.

(iv) It is clear from (i)–(iii) that $\mathcal{S}_{G,H}$ is Borel isomorphic to $\mathcal{S}_{S,T}/K$. Since the space $G/H=S/T$ is G-homogeneous, we see from equations (4.1)–(4.4) once again that the measure $\mu_{G,H}$ must be equivalent to a pseudo-image $\bar{\mu}_{S,T}$ of $\mu_{S,T}$ under the projection $\hat{S}\to\hat{S}/G$.

This completes the proof. We summarize as follows: we presume to know the spectral decomposition.

$$R_{S,T}=\int_{\mathcal{S}_{S,T}}^{\oplus}\gamma\,d\mu_{S,T}(\gamma).$$

Theorem 4.1 says that there is a map $\gamma\to\tau(\gamma)$ such that

$$R_{G,H}=\int_{\mathcal{S}_{S,T}/K}^{\oplus}\pi_{\gamma,\tau(\gamma)}\,d\bar{\mu}_{S,T}(\gamma).$$

To understand completely the latter decomposition we must describe the correspondence $\gamma\to\tau(\gamma)$. If S is abelian, it is easy: $\tau(\gamma)=1\times\gamma$ on $K_\gamma S$. What about non-abelian S? We can deduce two results by a closer inspection of the Mackey machinery. First since γ extends to a representation τ of $G_\gamma=K_\gamma S$, then all such extensions are obtained by multiplication by a character χ of K_γ, $\tau\otimes\chi$. If K_γ has no non-trivial characters, then $\tau(\gamma)$ is uniquely determined. But we can say more. According to (1.1), the support can only involve representations π which satisfy $\pi^\sigma\cong\bar{\pi}$. Now $\sigma\big|_{K_\gamma}\equiv\mathrm{Id}$. Since

$$\pi^{\sigma}_{\gamma,\tau(\gamma)}\cong\pi_{\gamma^\sigma,\tau(\gamma)^\sigma}\quad\text{and}\quad\bar{\pi}_{\gamma,\tau(\gamma)}\cong\pi_{\bar{\gamma},\bar{\tau}(\bar{\gamma})},$$

we have

$$K_{\bar{\gamma}}=K_{\gamma^\sigma}=(K_\gamma)^\sigma=K_\gamma.$$

Hence

$$\tau^\sigma \cong \bar{\tau}.$$

If we modify τ by a character χ of K_γ, then since $\sigma|_{K_\gamma} \equiv \mathrm{Id}$, we have $(\tau \otimes \chi)^\sigma = \tau^\sigma \otimes \chi$. But $\overline{\tau \otimes \chi} = \bar{\tau} \otimes \bar{\chi}$. It follows that $\chi = \bar{\chi}$. In particular the ambiguity in $\tau(\gamma)$ is only up to characters of order 2 of the group K_γ.

COROLLARY 4.2. *If for $\mu_{S,T}$-a.a. $\gamma \in \hat{S}$, the stability group K_γ supports no non-trivial character of order 2, then there exists a unique extension $\tau = \tau(\gamma)$ of γ to $G_\gamma = K_\gamma S$ such that $\tau^\sigma \cong \bar{\tau}$, and then*

$$R_{G,H} = \int_{\mathcal{S}_{S,T/K}} \pi_{\gamma,\tau(\gamma)} d\bar{\mu}_{S,T}(\gamma).$$

If the groups K_γ have characters of order 2, then the extension $\tau(\gamma)$ is not unambiguously specified. Nevertheless we can say somewhat more. Namely we assert that the Mackey obstruction for extending γ to a representation of the little group K_γ vanishes. Indeed the Mackey Machine [12] says there exists an extension of γ to a (likely) projective representation $\tilde{\gamma}$ of G_γ. A unique co-cycle class ω_γ is thus determined on K_γ. The representation $\tau = \tau(\gamma)$ must be of the form $\tau = \tilde{\gamma} \otimes \nu$, where ν is an irreducible projective representation of K_γ (with co-cycle $\bar{\omega}_\gamma$). But $\dim \tau = \dim \gamma \Rightarrow \dim \nu = 1$, i.e. ν is a projective character. Then τ itself qualifies as a candidate for $\tilde{\gamma}$. This can only happen if the co-cycle class ω_γ is trivial. We obtain

PROPOSITION 4.3. *For $\mu_{S,T}$-a.a. $\gamma \in \hat{S}$, the Mackey obstruction to extending γ to a representation of G_γ is trivial.*

Still there remains the ambiguity in the map $\gamma \to \tau(\gamma)$ up to characters of order 2. There is a natural conjecture I would like to formulate. It applies to most cases. By the general orbit method [4], [10], one has associated to each $\gamma \in \hat{S}$ an element $\theta \in \mathcal{AP}(S)$ which is an admissible, well-polarizable functional. Assume that generically, these are full (see [11]) — i.e. S_θ is connected, e.g. if S is exponential solvable. Then according to [11] (see also [3]), there exists a canonical extension $\tilde{\gamma}$ of γ to $\tilde{K}_\gamma$, the canonical 2-fold cover of K_γ. Assume now that S is simply connected solvable and almost every $\gamma \in \mathcal{S}_{S,T}$ corresponds to an element $\theta \in \mathcal{AP}(S)$ which has a K_γ-invariant polarization (e.g. if K is connected amenable. Indeed it should hold anyway since the Mackey obstruction vanishes). Then $\tilde{\gamma}$ passes to K_γ and I presume

CONJECTURE 4.4. $\tau(\gamma) = \tilde{\gamma}$.

EXAMPLE. Take S to be the Heisenberg group so that $\mathfrak{s}$ is generated by X, Y, Z with $[X, Y] = Z$. Select $K = \mathbf{Z}_2 = \{1, \varepsilon\}$ acting by $\varepsilon : X \to -X$, $Y \to -Y$, $Z \to Z$. The representations of $G = KS$ are $\pi_{t,\delta} = \delta \otimes \tilde{\gamma} \times \gamma$, where $\gamma = \gamma_t$ if γ has central character $\exp uZ \to e^{itu}$ and $\delta(\varepsilon) = \varepsilon^j$, $j = 1, 2$. Define an involution by $\sigma : X \to X$, $Y \to -Y$, $Z \to -Z$. Then $R_{G,H} = \int^{\otimes} \pi_{t,1} dt$. Note however that the representations $\pi_{t,\delta}$ also satisfy $\pi^\sigma \cong \bar{\pi}$.

We have also verified Conjecture 4.4 when $\mathfrak{s}$ is abelian. Now let us verify it when essentially no K_γ has an order 2 character. In fact to do that it suffices to establish

PROPOSITION 4.5. *The canonical representation $\tilde{\gamma}$ of K_γ satisfies $\tilde{\gamma}^\sigma \cong \bar{\tilde{\gamma}} \cong \tilde{\gamma}$.*

PROOF. The equivalence $\tilde{\gamma}^\sigma \cong \tilde{\gamma}$ is obvious. Now if $\gamma \in \hat{S}$, since K is connected amenable we can always choose $\theta \in \mathfrak{n}^*$ such that $\gamma = \gamma(\theta)$, θ is fixed by K_γ and has an invariant polarization $\mathfrak{b}$. Then the space of γ is realized (via holomorphic induction) on a space of functions

$$f : S \to \mathbf{C}, \qquad X * f = \omega(X) f, \qquad X \in \mathfrak{b} \qquad \text{etc.}$$

S acts by right translation (see e.g. [7, VI. A]). The representation $\bar{\gamma} = \gamma(-\theta)$ also can be realized by the K_γ-invariant polarization $\bar{\mathfrak{b}}$ and it is clear that the map $f \to \bar{f}$ is an intertwining operator for the action of K_γ. q.e.d.

For our last result we would like to extend the orbital description of the spectrum of $R_{G,H}$ outside the realm of nilpotent groups, where it is fact — or exponential solvable groups, where it is speculation — to that of semidirect products. In order to insure that we deal with fact rather than fiction, let us consider co-compact nilradical groups, i.e. $G = KN$ with N simply connected nilpotent normal and K compact. Then we can make use of [9]. The irreducible representations of G are parameterized in an orbital fashion by the fiber diagram

$$\begin{array}{c} \mathfrak{X}(G_\phi) \to \hat{G} \\ \qquad\quad \downarrow \\ \qquad\quad \mathfrak{A}(G)/G, \end{array}$$

where $\mathfrak{A}(G)$ is the allowable functionals in $\mathfrak{g}^*$, $\mathfrak{X}(G_\phi) = \{\tau \in \hat{G}_\phi : d\tau = i\phi \mid \mathfrak{g}_\phi\}$ is a finite fiber. We have an involutive automorphism σ of N, preserved by K, $M = N^\sigma$, $G = KN$. Suppose $\phi \in \mathfrak{A}(G)$ and $\theta = \phi \mid_{\mathfrak{n}}$. If $\gamma = \gamma(\theta) \in \hat{N}$, we have $G_\gamma = G_\theta N$ so that $G_\theta / N_\theta = G_\theta / N \cap G_\theta \cong G_\theta N / N = G_\gamma N / N = K_\gamma$ is compact. That is, any Levi component L of G_θ is compact. Any two such are N_θ-

conjugate. Now suppose $\phi(\mathfrak{l})=0$. Then I claim $\phi(\mathfrak{l}')=0$ for any Levi component of $\mathfrak{g}_\theta$. Indeed $\mathfrak{l}'=n\cdot\mathfrak{l}$, $n\in N_\theta$. Thus if $X\in\mathfrak{l}$, $n=\exp Y$, $Y\in\mathfrak{n}_\theta$ we have

$$\phi(n\cdot X)=\phi(e^{\operatorname{ad}Y}(X))=\phi(X)+\phi([Y,X])+\phi([Y[Y,X]])+\cdots=0$$

because $\phi(\mathfrak{l})=0$, $X\in\mathfrak{g}_\theta$ and $Y\in\mathfrak{n}_\theta$. Set

$$\mathcal{T}_{G,H}=\{\phi\in\mathfrak{A}(G):\theta=\phi\,|_{\mathfrak{n}},N\cdot\theta\cap\mathfrak{m}^\perp\neq 0,\phi(\mathfrak{l})=0\},$$

where $\mathfrak{l}$ is any Levi component of $\mathfrak{g}_\theta$. Two things are clear from all we have done: $\mathcal{T}_{G,H}$ is G-invariant and $\mathcal{S}_{G,H}$ projects bijectively onto $\mathcal{T}_{G,H}/G$ under the natural projection $\hat{G}\to\mathfrak{A}(G)/G$.

PROPOSITION 4.6. (i) *The natural map* $(\mathfrak{k}+\mathfrak{m})^\perp\to\mathfrak{g}^*\to\mathfrak{g}^*/G$ *is KM-equivariant.*

(ii) *The quotient map* $(\mathfrak{k}+\mathfrak{m})^\perp/KM\to\mathfrak{g}^*/G$ *is injective.*

(iii) CONJECTURE. *The image is precisely* $\mathcal{T}_{G,H}/G$.

PROOF. (i) This is obvious.

(ii) We must show that if ϕ_1, $\phi_2\in\mathfrak{g}^*$, $\phi_i(\mathfrak{k}+\mathfrak{m})=0$, $i=1,2$ and $g\cdot\phi_1=\phi_2$, then ϕ_1 and ϕ_2 are actually KM-conjugate. Write $g=kn$. Since the subspace $(\mathfrak{k}+\mathfrak{m})^\perp$ is preserved by K, it is no loss of generality to assume $k=1$. Now the automorphism σ differentiates to $\mathfrak{g}$, and its transpose ${}^t\sigma$ acts on $\mathfrak{g}^*$. If $\phi\in(\mathfrak{k}+\mathfrak{m})^\perp$, then ${}^t\sigma(\phi)=-\phi$. This is because ${}^t\sigma(\phi)\,|_{\mathfrak{n}}={}^t\sigma(\theta)=-\theta=-\phi\,|_{\mathfrak{n}}$ (see [1, 4.3]), and ${}^t\sigma(\phi)\,|_{\mathfrak{k}}=-\phi\,|_{\mathfrak{k}}=0$. Thus

$$-\phi_2={}^t\sigma(\phi_2)={}^t\sigma(n\cdot\phi_1)=\sigma(n)\cdot{}^t\sigma(\phi_1)=\sigma(n)(-\phi_1).$$

Therefore $\sigma(n)^{-1}n\in G_{\phi_1}$. But if we write $n=m\exp X$, $m\in M$, $X\in\mathfrak{q}=\{X\in\mathfrak{n}:\sigma(X)=-X\}$, then $\sigma(n)=m\exp -X$ and so $\sigma(n)^{-1}n=\exp 2X\in G_{\phi_1}$. But then $\exp X\in G_{\phi_1}$ and $n\cdot\phi_1=m\cdot\phi_1$. q.e.d.

(iii) I have not been able to prove this. If true, it would say that exactly as in Benoist's nilpotent situation, we have that $\mathcal{S}_{G,H}$ is parameterized by $\mathfrak{h}^\perp/H$.

5. Additional remarks

We provide here an answer to a question of L. Corwin. He asks: given N, a simply connected nilpotent Lie group, can one always produce a symmetric space N/M such that $R_{N,M}$ is hefty? The answer is no. Here is the example. The group N is the maximal nilpotent subgroup of the simple Lie group Sp$(2,1)$. It can be described via seven generators

$$\{X_i;1\leqq i\leqq 4\}\cup\{Z_j:1\leqq j\leqq 3\},$$

satisfying non-zero bracket relations

$$[X_1, X_2] = [X_4, X_3] = Z_1,$$

$$[X_1, X_3] = [X_2, X_4] = Z_2,$$

$$[X_1, X_4] = [X_3, X_2] = Z_3.$$

This is a nilpotent group with square-integrable representations. Suppose σ is an involutive automorphism with $M = N^\sigma$. We know (by Theorem 3.3) that $R_{N,M}$ is multiplicity-free. Can it be quasi-equivalent to R_N? The claim is no. Now σ passes to an involution of $\mathfrak{n}$, and as such it preserves the center $\mathfrak{z}$ of $\mathfrak{n}$. Consider the transformation $\sigma|_{\mathfrak{z}}$. The eigenvalues can only be ± 1. Suppose $+1$ is an eigenvalue. Then is $Z \in \mathfrak{z}$ such that $\sigma(Z) = Z$. In particular the one-parameter subgroup $\exp RZ$ lies in M. But then every representation in the spectrum $\mathscr{S}_{N,M}$ corresponds to a linear functional in the N-invariant Zariski-closed subvariety $(\mathbf{R}Z)^\perp$ in $\mathfrak{n}^*$. It follows from [13] that M cannot be hefty. Hence $\sigma|_{\mathfrak{z}} = -\mathrm{Id}$.

Next consider the involution induced by σ on $\mathfrak{n}/\mathfrak{z}$, i.e. $\sigma(X + \mathfrak{z}) = \sigma(X) + \mathfrak{z}$, $X \in \mathfrak{n}$. It too may only have eigenvalues ± 1. Moreover the matrix of the transformation $\sigma_{\mathfrak{n}/\mathfrak{z}}$ must be semisimple (since it has finite order). In particular, every generalized eigenvector is an ordinary eigenvector. Now we shall prove that there cannot be two independent eigenvectors whose eigenvalues have the same sign. That is, of course, preposterous since $\dim \mathfrak{n}/\mathfrak{z} = 4$. But indeed let $X + \mathfrak{z}$, $Y + \mathfrak{z}$, $X \notin \mathfrak{z}$, $Y \notin \mathfrak{z}$ be independent eigenvectors with eigenvalue ε ($\varepsilon = \pm 1$). Now $[X, Y] \in \mathfrak{z}$ (since $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$). But $\sigma|_{\mathfrak{z}} = -\mathrm{Id}$, therefore

$$-[X, Y] = \sigma[X, Y] = [\sigma X, \sigma Y] \in [\varepsilon X + \mathfrak{z}, \varepsilon Y + \mathfrak{z}] = \varepsilon^2[X, Y] = [X, Y].$$

This is impossible since — as it is easy to check from the commutation relations — $\forall W \in \mathfrak{n} \setminus \mathfrak{z}$, $Z_{\mathfrak{n}}(W) = \mathbf{R}W + \mathfrak{z}$. So for every involution σ of $\mathfrak{n}$, the transformation $\sigma|_{\mathfrak{z}}$ has a positive eigenvalue, and the corresponding symmetric space is slim.

Our final remark is the following. We have seen that the study of semidirect product symmetric spaces KS/KT may present interesting problems in determining their L^2 spectrum. Even when S is abelian and T is trivial, the spectra of $L^2(KS/K)$ may enjoy different properties depending on the constituents. Also, the techniques brought to bear in the study are varied and interesting — e.g. the orbit method, Mackey Machine, reciprocity, independence of induction results, and more. In fact the abelian symmetric space KS/K is a special case of a more general problem, in which little progress has been registered. Namely, describe the spectral decomposition of $L^2(G/K)$ for *any* semidirect product $G = KN$, N

normal. Naturally we assume N is type I and that $\hat{N}/G$ is smooth. We seek the spectral decomposition for the natural action of G in $L^2(N)$. All of the previously enumerated techniques have a role to play in this problem which I hope to address in a future publication.

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